

# HOLOMORPHIC KOSZUL-BRYLINSKI HOMOLOGY

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ABSTRACT. In this note, we study the Koszul-Brylinski homology of *holomorphic* Poisson manifolds. We show that it is isomorphic to the cohomology of a certain *smooth* complex Lie algebroid with values in the Evens-Lu-Weinstein duality module. As a consequence, we prove that the Evens-Lu-Weinstein pairing on Koszul-Brylinski homology is nondegenerate. Finally we compute the Koszul-Brylinski homology for Poisson structures on  $\mathbb{CP}^1 \times \mathbb{CP}^1$ .

## 1. INTRODUCTION

In [2], Brylinski introduced a homology theory for Poisson manifolds, which is nowadays called Koszul-Brylinski homology. Evens, Lu & Weinstein [7] and Xu [15] proved independently that, for unimodular Poisson manifolds, the Koszul-Brylinski homology is (up to a change of degree) isomorphic to the Lichnerowicz-Poisson cohomology [10]. And Evens, Lu & Weinstein introduced a pairing on Koszul-Brylinski homology groups. In this note, we study the Koszul-Brylinski homology of *holomorphic* Poisson manifolds. Koszul-Brylinski homology is defined as the hypercohomology of the complex of sheaves

$$\dots \xrightarrow{\partial_\pi} \Omega_X^{i+1} \xrightarrow{\partial_\pi} \Omega_X^i \xrightarrow{\partial_\pi} \Omega_X^{i-1} \xrightarrow{\partial_\pi} \dots,$$

where  $\partial_\pi = i_\pi \circ \partial - \partial \circ i_\pi$ . As is explained in [9], any holomorphic Poisson manifold gives rise to a holomorphic Lie algebroid structure  $(T_X)_\pi^*$  on the holomorphic vector bundle  $(T_X)^*$ , which in turn induces a complex Lie algebroid structure  $T_X^{0,1} \bowtie (T_X^{1,0})_\pi^*$  on the complex vector bundle  $T_X^{0,1} \oplus (T_X^{1,0})^*$ . We show that the cohomology of this complex Lie algebroid with values in the Evens-Lu-Weinstein duality module is isomorphic to the Koszul-Brylinski homology. As a consequence, we prove that the Evens-Lu-Weinstein pairing on Koszul-Brylinski homology is nondegenerate. We also introduce the Euler characteristic for the Koszul-Brylinski homology of a Poisson manifold and show that it coincides with the signed Euler characteristic of the manifold. Finally we compute the Koszul-Brylinski homology for Poisson structures on  $\mathbb{CP}^1 \times \mathbb{CP}^1$ . We refer the reader to the works of Etingof & Ginzburg [6] and Pichereau [13] for more on the Koszul-Brylinski homology of algebraic Poisson varieties.

## 2. HOLOMORPHIC LIE ALGEBROID COHOMOLOGY

Let  $A$  be a holomorphic Lie algebroid over a complex manifold  $X$ : i.e.  $A \rightarrow X$  is a holomorphic vector bundle whose sheaf of holomorphic sections  $\mathcal{A}$  is endowed with a

Lie bracket  $[\cdot, \cdot] : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ , and there exists a holomorphic bundle map  $A \xrightarrow{a} T_X$ , called anchor, which induces a morphism of sheaves of  $\mathcal{O}_X$ -modules  $\mathcal{A} \xrightarrow{a} \Theta_X$  such that

$$a([s_1, s_2]) = [a(s_1), a(s_2)], \quad \forall s_1, s_2 \in \mathcal{A}; \quad (1)$$

$$[s_1, fs_2] = (a(s_1)f)s_2 + f[s_1, s_2], \quad \forall s_1, s_2 \in \mathcal{A}, f \in \mathcal{O}_X. \quad (2)$$

This holomorphic Lie algebroid structure gives rise to a complex of sheaves:

$$\dots \xrightarrow{d_A} \Omega_A^{k-1} \xrightarrow{d_A} \Omega_A^k \xrightarrow{d_A} \Omega_A^{k+1} \xrightarrow{d_A} \dots,$$

where  $\Omega_A^k$  stands for the sheaf of holomorphic sections of the holomorphic vector bundle  $\wedge^k A^*$ , and  $d_A$  is given by the usual Cartan formula. By definition [7,9], the holomorphic Lie algebroid cohomology of  $A$  (with trivial coefficients) is the hypercohomology of this complex of sheaves:

$$H^*(A, \mathbb{C}) := \mathbb{H}^*(X, \Omega_A^\bullet).$$

A holomorphic vector bundle  $E \rightarrow X$  (with sheaf of holomorphic functions  $\mathcal{E}$ ) is said to be a module over the holomorphic Lie algebroid  $A$ , if there is a morphism of sheaves (of  $\mathbb{C}$ -modules)

$$\mathcal{A} \otimes \mathcal{E} \rightarrow \mathcal{E} : V \otimes s \mapsto \nabla_V s$$

such that, for any open subset  $U \subset X$ , the relations

$$\begin{aligned} \nabla_{fV} s &= f \nabla_V s \\ \nabla_V (fs) &= (\rho(V)f)s + f \nabla_V s \\ \nabla_V \nabla_W s - \nabla_W \nabla_V s &= \nabla_{[V,W]} s \end{aligned}$$

are satisfied  $\forall f \in \mathcal{O}_X(U)$ ,  $\forall V, W \in \mathcal{A}(U)$  and  $\forall s \in \mathcal{E}(U)$ . Such a morphism  $\nabla$  is called a representation of  $A$  on  $E$ . Given an  $A$ -module  $E \rightarrow X$ , one can form the complex of sheaves

$$\dots \xrightarrow{d_A^\nabla} \Omega_A^{k-1} \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{d_A^\nabla} \Omega_A^k \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{d_A^\nabla} \Omega_A^{k+1} \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{d_A^\nabla} \dots. \quad (3)$$

By definition, the Lie algebroid cohomology of  $A$  with values in  $E$  is the hypercohomology of this complex of sheaves:

$$H^*(A, E) := \mathbb{H}^*(X, \Omega_A^\bullet \otimes_{\mathcal{O}_X} \mathcal{E}).$$

Given a holomorphic Lie algebroid  $A$  with anchor  $a$ , we define  $a^{1,0} = \frac{1-iJ}{2} \circ a : A \rightarrow T_{\mathbb{C}}X$ . Here  $J$  stands for the almost complex structure  $J : T_X \rightarrow T_X$  of the complex manifold  $X$ . Of course, for any holomorphic function  $f \in \mathcal{O}_X(U)$ , we have  $a^{1,0}(V)f = a(V)f$ , for all  $V \in \Gamma(U, A)$ . Now regard  $A$  as a complex vector bundle. The Lie bracket, which was defined so far only on the sheaf of holomorphic sections of  $A$ , extends naturally to all smooth sections through the Leibniz rule

$$[s_1, fs_2] = (a^{1,0}(s_1)f)s_2 + f[s_1, s_2], \quad \forall s_1, s_2 \in \Gamma(A), f \in C^\infty(X, \mathbb{C}),$$

with  $a^{1,0}$  substituted to  $a$ . We use the symbol  $A^{1,0}$  to denote the resulting complex Lie algebroid structure on  $A$  [9].

Now recall that the complex vector bundle  $T_X^{0,1}$  is endowed with a canonical complex Lie algebroid structure whose Lie bracket is completely determined by the relation  $[\partial_{\bar{z}_j}, \partial_{\bar{z}_k}] = 0$  and the anchor, which is simply the injection  $T_X^{0,1} \hookrightarrow T_X \otimes \mathbb{C}$ .

**Proposition 2.1** ([9, Theorems 4.2 and 4.8]). *If  $A$  is a holomorphic vector bundle with anchor  $a$  over a complex manifold  $X$ , there exists a unique complex Lie algebroid structure on the complex vector bundle  $T_X^{0,1} \oplus A^{1,0}$  with anchor  $a^\bowtie(X^{0,1} + \xi) = X^{0,1} + a^{1,0}(\xi)$  such that  $[\bar{\Theta}_X, \mathcal{A}] = 0$  and both  $T_X^{0,1}$  and  $A^{1,0}$  are Lie subalgebroids.*

This complex Lie algebroid is denoted  $T_X^{0,1} \bowtie A^{1,0}$ . The pair  $(T_X^{0,1}, A^{1,0})$  is an example of matched pair [9, 11, 12].

**Theorem 2.2** ([9, Lemma 4.16 and Theorem 4.19]). *Let  $A \rightarrow X$  be a holomorphic Lie algebroid and  $E \rightarrow X$  a complex vector bundle. Then  $E$  is a module over the holomorphic Lie algebroid  $A$  if, and only if,  $E$  is a module over the complex Lie algebroid  $T_X^{0,1} \bowtie A^{1,0}$ . Moreover, we have*

$$H^*(A, E) \cong H^*(T_X^{0,1} \bowtie A^{1,0}, E).$$

Note that the complex Lie algebroid  $T_X^{0,1} \bowtie A^{1,0}$  is an elliptic Lie algebroid in the sense of Block [1]. That is,  $\Re \circ a^\bowtie$  is surjective. Therefore, when  $X$  is compact, the cohomology groups  $H^*(T_X^{0,1} \bowtie A^{1,0}, E)$  are finite dimensional and we can consider the Euler characteristic

$$\chi(A, E) = \sum_i (-1)^i \dim H^i(A, E). \quad (4)$$

**Proposition 2.3.** *Let  $A \rightarrow X$  be a holomorphic Lie algebroid and  $E$  an  $A$ -module. Assume that  $X$  is compact. Then*

$$\chi(A, E) = \sum_i (-1)^i \chi(X, \wedge^i A^* \otimes E),$$

where  $\chi(X, \wedge^i A^* \otimes E)$  is the Euler characteristic of the holomorphic bundle  $\wedge^i A^* \otimes E$ .

*Proof.* By definition,  $H^*(A, E)$  is isomorphic to the hypercohomology  $\mathbb{H}^*(X, \Omega_A^\bullet \otimes_{\mathcal{O}_X} \mathcal{E})$  of the complex of sheaves (3), which, according to Theorem 2.2, is computed by the total cohomology  $H^n(T_X^{0,1} \bowtie A^{1,0}, E)$  of the double complex

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & \\ & \uparrow d_{A^{1,0}}^\nabla & & \uparrow d_{A^{1,0}}^\nabla & & \uparrow d_{A^{1,0}}^\nabla & \\ \Omega_X^{0,0} \otimes C_X^\infty & \xrightarrow{\bar{\partial}} \mathcal{A}^{2,0} & \xrightarrow{\bar{\partial}} & \Omega_X^{0,1} \otimes C_X^\infty & \xrightarrow{\bar{\partial}} \mathcal{A}^{2,0} & \xrightarrow{\bar{\partial}} & \Omega_X^{0,2} \otimes C_X^\infty & \xrightarrow{\bar{\partial}} \mathcal{A}^{2,\bar{\partial}} \longrightarrow \dots \\ & \uparrow d_{A^{1,0}}^\nabla & & \uparrow d_{A^{1,0}}^\nabla & & \uparrow d_{A^{1,0}}^\nabla & \\ \Omega_X^{0,0} \otimes C_X^\infty & \xrightarrow{\bar{\partial}} \mathcal{A}^{1,0} & \xrightarrow{\bar{\partial}} & \Omega_X^{0,1} \otimes C_X^\infty & \xrightarrow{\bar{\partial}} \mathcal{A}^{1,0} & \xrightarrow{\bar{\partial}} & \Omega_X^{0,2} \otimes C_X^\infty & \xrightarrow{\bar{\partial}} \mathcal{A}^{1,\bar{\partial}} \longrightarrow \dots \\ & \uparrow d_{A^{1,0}}^\nabla & & \uparrow d_{A^{1,0}}^\nabla & & \uparrow d_{A^{1,0}}^\nabla & \\ \Omega_X^{0,0} \otimes C_X^\infty & \xrightarrow{\bar{\partial}} \mathcal{A}^{0,0} & \xrightarrow{\bar{\partial}} & \Omega_X^{0,1} \otimes C_X^\infty & \xrightarrow{\bar{\partial}} \mathcal{A}^{0,0} & \xrightarrow{\bar{\partial}} & \Omega_X^{0,2} \otimes C_X^\infty & \xrightarrow{\bar{\partial}} \mathcal{A}^{0,\bar{\partial}} \longrightarrow \dots \end{array}$$

where  $\Omega_X^{i,j} = \Gamma(\wedge^i T_X^{1,0} \otimes \wedge^j T_X^{0,1})$  and  $\mathcal{A}^{k,l} = \Gamma(\wedge^k (A^{1,0})^* \otimes \wedge^l (A^{0,1})^* \otimes E)$ .

Set  $C^{p,q} = \Omega_X^{0,p} \otimes_{C_X^\infty} \mathcal{A}^{q,0}$  and  $C^n = \bigoplus_{p+q=n} C^{p,q}$ . The spectral sequence induced by the filtration  $F_q(C^n) = \bigoplus_{\substack{\tilde{q} \geq q \\ \tilde{p} + \tilde{q} = n}} C^{\tilde{p}, \tilde{q}}$  of  $C^\bullet$  starts with  $E_0^{p,q} = C^{p,q}$ ,  $d_0^{p,q} = \bar{\partial}$  and  $E_1^{p,q} = H^p(C^{\bullet,q}, \bar{\partial})$ , and converges to  $H^n(T_X^{0,1} \bowtie A^{1,0}, E)$ .

Since the Euler characteristic of  $E_r^{p,q}$  does not change from one sheet to the next, we have

$$\begin{aligned}
\chi(A, E) &= \sum_n (-1)^n \dim H^n(A, E) \\
&= \sum_n (-1)^n \dim H^n(T_X^{0,1} \bowtie A^{1,0}, E) \\
&= \sum_n (-1)^n \dim \left( \bigoplus_{p+q=n} E_\infty^{p,q} \right) \\
&= \sum_n (-1)^n \dim \left( \bigoplus_{p+q=n} E_1^{p,q} \right) \\
&= \sum_n (-1)^n \dim \left( \bigoplus_{p+q=n} H^p(C^{\bullet,q}, \bar{\partial}) \right) \\
&= \sum_q (-1)^q \left( \sum_p (-1)^p \dim H^p(C^{\bullet,q}, \bar{\partial}) \right) \\
&= \sum_q (-1)^q \chi(X, \wedge^q A^* \otimes E). \quad \square
\end{aligned}$$

### 3. HOLOMORPHIC POISSON MANIFOLDS

A holomorphic Poisson manifold is a complex manifold  $X$  whose sheaf of holomorphic functions  $\mathcal{O}_X$  is a sheaf of Poisson algebras. By a sheaf of Poisson algebras over  $X$ , we mean that, for each open subset  $U \subset X$ , the ring  $\mathcal{O}_X(U)$  is endowed with a Poisson bracket such that all restriction maps  $\mathcal{O}_X(U) \rightarrow \mathcal{O}_X(V)$  (for arbitrary open subsets  $V \subset U \subset X$ ) are morphisms of Poisson algebras. Moreover, given an open subset  $U \subset X$ , an open covering  $\{U_i\}_{i \in I}$  of  $U$ , and a pair of functions  $f, g \in \mathcal{O}_X(U)$ , the local data  $\{f|_{U_i}, g|_{U_i}\}$  ( $i \in I$ ) glue up and give  $\{f|_U, g|_U\}$  if they coincide on the overlaps  $U_i \cap U_j$ . On a given complex manifold  $X$ , the holomorphic Poisson structures are in one-to-one correspondence with the sections  $\pi \in \Gamma(\wedge^2 T_X^{1,0})$  such that  $\bar{\partial}\pi = 0$  and  $[\pi, \pi] = 0$ . The Poisson bracket on functions and the bivector field are related by the formula  $\pi(\partial f, \partial g) = \{f, g\}$ , where  $f, g \in \mathcal{O}_X$ .

Given a holomorphic Poisson bracket

$$\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X \rightarrow \mathcal{O}_X : (f, g) \mapsto \{f, g\},$$

the formula

$$[f_1 dg_1, f_2 dg_2] = f_1 X_{g_1}(f_2) dg_2 - f_2 X_{g_2}(f_1) dg_1 + f_1 f_2 d\{g_1, g_2\}, \quad (5)$$

where  $f_1, f_2, g_1, g_2 \in \mathcal{O}_X$ , defines a Lie bracket on  $\Omega_X$ . Here  $X_f \in \Theta_X$  denotes the derivation

$$X_f : \mathcal{O}_X \rightarrow \mathcal{O}_X : g \mapsto \{f, g\}$$

of  $\mathcal{O}_X$  associated to the holomorphic function  $f \in \mathcal{O}_X$ . Since  $\Gamma((T_X^{1,0})^*) = C^\infty(X, \mathbb{C})\Omega_X$ , the bracket on  $\Omega_X$  extends to  $\Gamma((T_X^{1,0})^*)$  by the Leibniz rule:

$$[fdz_k, gdz_l] = fX_{z_k}(g)dz_l - gX_{z_l}(f)dz_k + fg d\{z_k, z_l\},$$

for all  $f, g \in C^\infty(X, \mathbb{C})$ . If the bivector field associated to the Poisson bracket on  $\mathcal{O}_X$  is  $\pi \in \Theta_X^2 \subset \Gamma(\wedge^2 T_X^{1,0})$ , then the Lie bracket is given by

$$[\alpha, \beta] = L_{\pi^\sharp \alpha} \beta - L_{\pi^\sharp \beta} \alpha - \partial(\pi(\alpha, \beta)), \quad \forall \alpha, \beta \in \Gamma((T_X^{1,0})^*).$$

Once its sheaf of sections  $\Omega_X$  has been endowed with this Lie bracket, the cotangent bundle  $(T_X)^*$  becomes a holomorphic Lie algebroid with anchor map  $\pi^\sharp : (T_X)^* \rightarrow T_X$ , which we refer to by the symbol  $(T_X)_\pi^*$ . By Proposition 2.1, we can associate to it the complex Lie algebroid  $T_X^{0,1} \bowtie (T_X^{1,0})_\pi^*$ .

The complex Lie algebroid structure on  $T_X^{0,1} \bowtie (T_X^{1,0})_\pi^*$  is characterized as follows: the anchor is  $\text{id}_{T_X^{0,1}} \oplus \pi^\sharp : T_X^{0,1} \oplus (T_X^{1,0})^* \rightarrow T_X \otimes \mathbb{C}$ , and the Lie bracket on  $\Gamma(T_X^{0,1} \bowtie (T_X^{1,0})_\pi^*)$  satisfies  $[\overline{\Theta}_X, \Omega_X] = 0$ , coincides with the Lie bracket of vector fields on  $\overline{\Theta}_X$  and with the bracket defined by (5) on  $\Omega_X$  [9].

#### 4. HOLOMORPHIC KOSZUL-BRYLINSKI HOMOLOGY

Let  $\Theta_X^k$  and  $\Omega_X^k$  denote the sheaves of holomorphic sections of  $\wedge^k T_X$  and  $\wedge^k (T_X)^*$ , respectively.

The Koszul-Brylinski operator  $\partial_\pi : \Omega_X^k \rightarrow \Omega_X^{k-1}$  is defined as  $\partial_\pi := \iota_\pi \partial - \partial \iota_\pi$ , where  $\partial : \Omega_X^k \rightarrow \Omega_X^{k+1}$  is the holomorphic exterior differential (i.e. the Dolbeault operator) and  $\iota_\pi : \Omega_X^k \rightarrow \Omega_X^{k-2}$  is the contraction with the holomorphic Poisson bivector field  $\pi$  [2, 8]. The operator  $\partial_\pi$  satisfies  $\partial_\pi^2 = 0$ ,  $\partial_\pi d + d\partial_\pi = 0$ , and

$$\partial_\pi(\alpha \wedge \beta) = \partial_\pi \alpha \wedge \beta + (-1)^k \alpha \wedge \partial_\pi \beta + (-1)^k [\alpha, \beta], \quad \forall \alpha \in \Omega_X^k, \beta \in \Omega_X^l.$$

**Definition 4.1.** *Let  $(X, \pi)$  be a holomorphic Poisson manifold. Its Koszul-Brylinski homology is the hypercohomology of the complex of sheaves*

$$\dots \xrightarrow{\partial_\pi} \Omega_X^{k+1} \xrightarrow{\partial_\pi} \Omega_X^k \xrightarrow{\partial_\pi} \Omega_X^{k-1} \xrightarrow{\partial_\pi} \dots \quad (6)$$

which is denoted  $H_*(X, \pi)$ .

**Remark 4.2.** *If  $\pi = 0$ , we have  $H_k(X, \pi) \cong \bigoplus_{j-i=n-k} H^j(X, \Omega_X^i)$ .*

As was pointed out earlier, a holomorphic Poisson manifold  $(X, \pi)$  automatically gives rise to a holomorphic Lie algebroid structure  $(T_X)_\pi^*$ . The Lichnerowicz-Poisson cohomology  $H^*(X, \pi; E)$  of  $(X, \pi)$  with coefficients in a  $(T_X)_\pi^*$ -module  $E$  is defined to be the Lie algebroid cohomology of  $(T_X)_\pi^*$  with coefficients in the module  $E$ , i.e. the hypercohomology of the complex of sheaves

$$\dots \xrightarrow{d_\pi^\nabla} \Theta_X^{k-1} \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{d_\pi^\nabla} \Theta_X^k \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{d_\pi^\nabla} \Theta_X^{k+1} \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{d_\pi^\nabla} \dots$$

In particular, when  $E$  is the trivial module  $X \times \mathbb{C} \rightarrow X$ , the associated differential complex is

$$\dots \xrightarrow{d_\pi} \Theta_X^{k-1} \xrightarrow{d_\pi} \Theta_X^k \xrightarrow{d_\pi} \Theta_X^{k+1} \xrightarrow{d_\pi} \dots$$

One has  $d_\pi V = [\pi, V]$ . The hypercohomology of this complex of sheaves is the holomorphic Lichnerowicz-Poisson cohomology  $H^*(X, \pi)$  of the holomorphic Poisson manifold  $(X, \pi)$  [9].

Assuming  $X$  compact, let

$$\chi^{LP}(X, \pi; E) = \sum_i (-1)^i \dim H^i(X, \pi; E) \quad (7)$$

be the Euler characteristic of the Lichnerowicz-Poisson cohomology  $H^*(X, \pi; E)$ .

**Proposition 4.3.** *If  $(X, \pi)$  is a compact holomorphic Poisson manifold, then*

$$\chi^{LP}(X, \pi; E) = \sum_i (-1)^i \chi(X, \wedge^i T_X \otimes E),$$

where  $\chi(X, \wedge^i T_X \otimes E)$  stands for the usual Euler characteristic of the holomorphic bundle  $\wedge^i T_X \otimes E$ .

*Proof.* By definition, we have  $H^k(X, \pi; E) = H^k((T_X)_\pi^*, \wedge^n(T_X)^*)$ , whence

$$\chi^{LP}(X, \pi; E) = \chi((T_X)_\pi^*, \wedge^n(T_X)^*).$$

Therefore, it suffices to apply Proposition 2.3 to the Lie algebroid  $A = (T_X)_\pi^*$  and its module  $E = \wedge^n(T_X)^*$  to conclude.  $\square$

A result of Evens, Lu & Weinstein (transposed to the holomorphic setting) asserts that, if  $A \rightarrow X$  is a holomorphic Lie algebroid with  $\dim_{\mathbb{C}} X = n$  and  $\text{rk}_{\mathbb{C}} A = r$ , the holomorphic vector bundle  $Q_A = \wedge^r A \otimes \wedge^n(T_X)^*$  is naturally a module over  $A$ . When the holomorphic Lie algebroid  $A$  is the cotangent bundle  $(T_X)_\pi^*$  of a holomorphic Poisson manifold  $(X, \pi)$ , we have  $Q_A = \wedge^n(T_X)^* \otimes \wedge^n(T_X)^*$ . Its square root  $\sqrt{Q_A} = \wedge^n(T_X)^*$  is also an  $A$ -module; the representation is the map

$$\Omega_X \otimes \Omega_X^n \rightarrow \Omega_X^n : \alpha \otimes \omega \mapsto \nabla_\alpha \omega$$

such that  $\nabla_{df} \omega = L_{X_f} \omega$ , for all  $f \in \mathcal{O}_X$  and  $\omega \in \Omega_X^n$ . Here  $\Omega_X$  and  $\Omega_X^n$  are the sheaves of holomorphic sections of  $(T_X)^*$  and  $\wedge^n(T_X)^*$  respectively. Hence, we obtain the complex of sheaves

$$\dots \xrightarrow{d_\pi^\nabla} \Theta_X^{k-1} \otimes_{\mathcal{O}_X} \Omega_X^n \xrightarrow{d_\pi^\nabla} \Theta_X^k \otimes_{\mathcal{O}_X} \Omega_X^n \xrightarrow{d_\pi^\nabla} \Theta_X^{k+1} \otimes_{\mathcal{O}_X} \Omega_X^n \xrightarrow{d_\pi^\nabla} \dots \quad (8)$$

An argument of Evens, Lu & Weinstein (see [7, Equation (22)]) adapted to the holomorphic context shows that the isomorphism of sheaves of  $\mathcal{O}_X$ -modules

$$\tau : \Theta_X^k \otimes_{\mathcal{O}_X} \Omega_X^n \rightarrow \Omega_X^{n-k} : X \otimes \alpha \mapsto \iota_X \alpha$$

is in fact an isomorphism between the complexes of sheaves (8) and (6):

$$\begin{array}{ccccccc}
\Omega_X^n & \longrightarrow & \cdots & \longrightarrow & \Theta_X^k \otimes_{\mathcal{O}_X} \Omega_X^n & \xrightarrow{d_\pi^\nabla} & \Theta_X^{k+1} \otimes_{\mathcal{O}_X} \Omega_X^n & \longrightarrow & \cdots & \longrightarrow & \Theta_X^n \otimes_{\mathcal{O}_X} \Omega_X^n & (9) \\
\text{id} \downarrow & & & & \tau \downarrow & & \tau \downarrow & & & & & \tau \downarrow \\
\Omega_X^n & \longrightarrow & \cdots & \longrightarrow & \Omega_X^{n-k} & \xrightarrow{(-1)^{k+1} \partial_\pi} & \Omega_X^{n-k-1} & \longrightarrow & \cdots & \longrightarrow & \Omega_X^0
\end{array}$$

This isomorphism of complexes of sheaves induces an isomorphism of the corresponding sheaf cohomologies. Thus we obtain the following theorem, which is a holomorphic analogue of a result of Evens, Lu & Weinstein [7, Corollary 4.6].

**Theorem 4.4.** *For any holomorphic Poisson manifold  $(X, \pi)$ , the chain map  $\tau$  induces an isomorphism*

$$H^k(X, \pi; \wedge^n(T_X)^*) \xrightarrow{\cong} H_{2n-k}(X, \pi).$$

Assume that  $(X, \pi)$  is a compact holomorphic Poisson manifold. Let

$$\chi_{KB}(X, \pi) = \sum_i (-1)^i \dim H_i(X, \pi) \quad (10)$$

be the Euler characteristic of the Koszul-Brylinski homology.

**Theorem 4.5.** *For a compact holomorphic Poisson manifold  $(X, \pi)$ , we have*

$$\chi_{KB}(X, \pi) = (-1)^n \chi(X),$$

where  $\chi(X)$  denotes the standard Euler characteristic of  $X$ .

*Proof.* We have

$$\begin{aligned}
\chi_{KB}(X, \pi) &= \chi^{LP}(X, \pi; \wedge^n(T_X)^*) && \text{by Theorem 4.4} \\
&= \sum_i (-1)^i \chi(X, \wedge^i T_X \otimes \wedge^n(T_X)^*) && \text{by Proposition 4.3} \\
&= (-1)^n \sum_j (-1)^j \chi(X, \wedge^j(T_X)^*) \\
&= (-1)^n \chi(T_X, \mathbb{C}) && \text{by Proposition 2.3.}
\end{aligned}$$

Of course, since  $\Omega_X^\bullet \xrightarrow{\partial} \Omega_X^{\bullet+1}$  and  $\Gamma(\wedge^\bullet(T_X \otimes \mathbb{C})^*) \xrightarrow{d} \Gamma(\wedge^{\bullet+1}(T_X \otimes \mathbb{C})^*)$  are two acyclic resolutions of the locally constant sheaf  $\mathbb{C}$  over  $X$ , we have

$$\begin{aligned}
\chi(T_X, \mathbb{C}) &= \sum_i (-1)^i \dim H^i(\Gamma(\Omega_X^\bullet) \xrightarrow{\partial} \Gamma(\Omega_X^{\bullet+1})) \\
&= \sum_i (-1)^i \dim H^i(\Gamma(\wedge^\bullet(T_X \otimes \mathbb{C})^*) \xrightarrow{d} \Gamma(\wedge^{\bullet+1}(T_X \otimes \mathbb{C})^*)) = \chi(X). \quad \square
\end{aligned}$$

**Definition 4.6.** *A holomorphic Poisson manifold  $(X, \pi)$  is said to be unimodular if  $\wedge^n(T_X)^*$  is isomorphic, as a  $(T_X)_\pi^*$ -module, to the trivial module  $\mathbb{C}$ .*

The notion of modular class was introduced independently by Brylinski & Zuckerman [3] for holomorphic Poisson manifolds, and by Weinstein [14] for real Poisson manifolds. For the relation between Calabi-Yau algebras and unimodular Poisson structures, see [4].

From the definition, it is clear that a holomorphic Poisson manifold  $(X, \pi)$  is unimodular if and only if there exists a global holomorphic section  $\omega \in \Omega_X^n$  such that the vector field  $H \in \Theta_X$  defined by

$$\nabla_{df}\omega = L_{X_f}\omega = H(f) \cdot \omega \quad (f \in \mathcal{O}_X)$$

is a holomorphic Hamiltonian vector field.

**Proposition 4.7.** *For a unimodular holomorphic Poisson manifold  $(X, \pi)$ , the chain map  $\tau$  induces an isomorphism*

$$H^k(X, \pi) \xrightarrow{\cong} H_{2n-k}(X, \pi).$$

## 5. KOSZUL-BRYLINSKI DOUBLE COMPLEX

In this section, we describe a double complex computing the Koszul-Brylinski homology.

**Theorem 5.1.** *The Koszul-Brylinski homology of a holomorphic Poisson manifold  $(X, \pi)$  is isomorphic to the total cohomology of the double complex*

$$\begin{array}{ccccccc} \dots & \longrightarrow & \Omega_X^{n-k+1,0} & \xrightarrow{(-1)^k \partial_\pi} & \Omega_X^{n-k,0} & \xrightarrow{(-1)^{k+1} \partial_\pi} & \Omega_X^{n-k-1,0} \longrightarrow \dots \\ & & \bar{\partial} \downarrow & & \bar{\partial} \downarrow & & \bar{\partial} \downarrow \\ \dots & \longrightarrow & \Omega_X^{n-k+1,1} & \xrightarrow{(\Gamma^1)^{1+k} \partial_\pi} & \Omega_X^{n-k,1} & \xrightarrow{(\Gamma^1)^{1+k+1} \partial_\pi} & \Omega_X^{n-k-1,1} \longrightarrow \dots \\ & & \bar{\partial} \downarrow & & \bar{\partial} \downarrow & & \bar{\partial} \downarrow \\ \dots & \longrightarrow & \Omega_X^{n-k+1,2} & \xrightarrow{(\bar{\Gamma}^1)^{2+k} \partial_\pi} & \Omega_X^{n-k,2} & \xrightarrow{(\bar{\Gamma}^1)^{2+k+1} \partial_\pi} & \Omega_X^{n-k-1,2} \longrightarrow \dots \\ & & \bar{\partial} \downarrow & & \bar{\partial} \downarrow & & \bar{\partial} \downarrow \\ & & \vdots & & \vdots & & \vdots \end{array}$$

*Proof.* According to Theorem 2.2, we have

$$H_*(X, \pi) \cong H^*(T_X^{0,1} \bowtie (T_X^{1,0})_\pi^*, \wedge^n (T_X^{1,0})^*).$$

The r.h.s. is the Lie algebroid cohomology of  $T_X^{0,1} \bowtie (T_X^{1,0})_\pi^*$  with coefficients in the module  $\wedge^n (T_X^{1,0})^*$ . Moreover, the representation of the complex Lie algebroid  $T_X^{0,1} \bowtie (T_X^{1,0})_\pi^*$  on  $\wedge^n (T_X^{1,0})^*$  is the map

$$\Gamma(T_X^{0,1} \oplus (T_X^{1,0})^*) \otimes \Gamma(\wedge^n (T_X^{1,0})^*) \rightarrow \Gamma(\wedge^n (T_X^{1,0})^*) : (X + \xi, \omega) \mapsto \nabla_{X+\xi}\omega$$

defined by

$$\begin{aligned} \nabla_{\partial_{\bar{z}_k}}(f dz_1 \wedge \dots \wedge dz_n) &= \frac{\partial f}{\partial \bar{z}_k} dz_1 \wedge \dots \wedge dz_n \\ \nabla_{dz_l}(f dz_1 \wedge \dots \wedge dz_n) &= L_{X_{z_l}}(f dz_1 \wedge \dots \wedge dz_n) \end{aligned}$$



(for all  $f \in C^\infty(X, \mathbb{C})$ ).

Consider the complex

$$\Gamma\left(\wedge^m (T_X^{0,1} \oplus (T_X^{1,0})^*)^* \otimes \wedge^n (T_X^{1,0})^*\right) \xrightarrow{d_{\bowtie}^\nabla} \Gamma\left(\wedge^{m+1} (T_X^{0,1} \oplus (T_X^{1,0})^*)^* \otimes \wedge^n (T_X^{1,0})^*\right) \quad (11)$$

Set  $C^{k,l} = \wedge^k (T_X^{0,1})^* \otimes \wedge^l T_X^{1,0} \otimes \wedge^n (T_X^{1,0})^*$  so that

$$\wedge^m (T_X^{0,1} \oplus (T_X^{1,0})^*)^* \otimes \wedge^n (T_X^{1,0})^* = \bigoplus_{k+l=m} C^{k,l}.$$

Since  $A := T_X^{0,1}$  and  $B := (T_X^{1,0})_\pi^*$  are complex Lie subalgebroids of  $T_X^{0,1} \bowtie (T_X^{1,0})_\pi^*$ , one has

$$d_{\bowtie}^\nabla \Gamma(C^{k,l}) \subset \Gamma(C^{k+1,l} \oplus C^{k,l+1}).$$

Composing  $d_{\bowtie}^\nabla$  with the natural projections on each of the direct summands, we get the commutative diagram

$$\begin{array}{ccccc} & & \Gamma(C^{k,l}) & & \\ & \swarrow \partial_A^\nabla & \downarrow d_{\bowtie}^\nabla & \searrow (-1)^k \partial_B^\nabla & \\ \Gamma(C^{k+1,l}) & \longleftarrow & \Gamma(C^{k+1,l} \oplus C^{k,l+1}) & \longrightarrow & \Gamma(C^{k,l+1}), \end{array}$$

where the operators  $\partial_A^\nabla$  and  $\partial_B^\nabla$  are given by

$$\begin{aligned} & (\partial_A^\nabla \alpha)(A_0, \dots, A_k, B_1, \dots, B_l) \\ &= \sum_{i=0}^k (-1)^i \left( \nabla_{A_i}(\alpha(A_0, \dots, \widehat{A_i}, \dots, A_k, B_1, \dots, B_l)) \right. \\ & \quad \left. - \sum_{j=1}^l \alpha(A_0, \dots, \widehat{A_i}, \dots, A_k, B_1, \dots, \text{pr}_B[A_i, B_j], \dots, B_l) \right) \\ & \quad + \sum_{i < j} (-1)^{i+j} \alpha([A_i, A_j], A_0, \dots, \widehat{A_i}, \dots, \widehat{A_j}, \dots, A_k, B_1, \dots, B_l) \end{aligned} \quad (12)$$

and

$$\begin{aligned} & (\partial_B^\nabla \alpha)(A_1, \dots, A_k, B_0, \dots, B_l) \\ &= \sum_{i=0}^l (-1)^i \left( \nabla_{B_i}(\alpha(A_1, \dots, A_k, B_0, \dots, \widehat{B_i}, \dots, B_l)) \right. \\ & \quad \left. - \sum_{j=1}^k \alpha(A_1, \dots, \text{pr}_A[B_i, A_j], \dots, A_k, B_0, \dots, \widehat{B_i}, \dots, B_l) \right) \\ & \quad + \sum_{i < j} (-1)^{i+j} \alpha(A_1, \dots, A_k, [B_i, B_j], B_0, \dots, \widehat{B_i}, \dots, \widehat{B_j}, \dots, B_l), \end{aligned} \quad (13)$$

for all  $\alpha \in \Gamma(\wedge^k A^* \otimes \wedge^l B^*)$ ,  $A_0, \dots, A_k \in \Gamma(A)$  and  $B_0, \dots, B_k \in \Gamma(B)$ . Here  $\text{pr}_B[A_i, B_j]$  denotes the  $B$ -component of  $[A_i, B_j] \in A \bowtie B$  and  $\text{pr}_A[B_i, A_j]$  the  $A$ -component of  $[B_i, A_j]$ .

Since  $d_{\bowtie}^\nabla = \partial_A^\nabla + (-1)^k \partial_B^\nabla$ , it follows from  $(d_{\bowtie}^\nabla)^2 = 0$  that  $(\partial_A^\nabla)^2 = 0$ ,  $(\partial_B^\nabla)^2 = 0$  and  $\partial_A^\nabla \circ \partial_B^\nabla = \partial_B^\nabla \circ \partial_A^\nabla$ . Thus the complex (11) is the total complex of the double complex

$$\begin{array}{ccc} \Gamma(C^{k,l}) & \xrightarrow{\partial_B^\nabla} & \Gamma(C^{k,l+1}) \\ \partial_A^\nabla \downarrow & & \downarrow \partial_A^\nabla \\ \Gamma(C^{k+1,l}) & \xrightarrow{\partial_B^\nabla} & \Gamma(C^{k+1,l+1}) \end{array}$$

Hence it follows that  $H^*(T_X^{0,1} \bowtie (T_X^{1,0})_\pi^*, \wedge^n(T_X^{1,0})^*)$  is isomorphic to the total cohomology of the double complex

$$\begin{array}{ccc} \Gamma(\wedge^i(T_X^{0,1})^* \otimes \wedge^j(T_X^{1,0}) \otimes \wedge^n(T_X^{1,0})^*) & \xrightarrow{\partial_B^\nabla} & \Gamma(\wedge^i(T_X^{0,1})^* \otimes \wedge^{j+1}(T_X^{1,0}) \otimes \wedge^n(T_X^{1,0})^*) \\ \partial_A^\nabla \downarrow & & \downarrow \partial_A^\nabla \\ \Gamma(\wedge^{i+1}(T_X^{0,1})^* \otimes \wedge^j(T_X^{1,0}) \otimes \wedge^n(T_X^{1,0})^*) & \xrightarrow{\partial_B^\nabla} & \Gamma(\wedge^{i+1}(T_X^{0,1})^* \otimes \wedge^{j+1}(T_X^{1,0}) \otimes \wedge^n(T_X^{1,0})^*) \end{array}$$

By  $\tau$  we denote the natural contraction map

$$\tau : \Gamma((\wedge^i(T_X^{0,1})^* \otimes \wedge^j(T_X^{1,0}))^* \otimes \wedge^n(T_X^{1,0})^*) \rightarrow \Omega^{n-j,i}, \quad (14)$$

which is an isomorphism of  $C^\infty(X, \mathbb{C})$ -modules.

Take a local holomorphic chart  $(U; z_1, \dots, z_n)$  of  $X$ , and set

$$\begin{aligned} b &= \partial_{z_{j_1}} \wedge \dots \wedge \partial_{z_{j_l}}; \\ \omega &= dz_1 \wedge \dots \wedge dz_n. \end{aligned}$$

Because of (9), we have

$$\tau d_\pi^\nabla(b \otimes \omega) = (-1)^{l+1} \partial_\pi \tau(b \otimes \omega). \quad (15)$$

**Lemma 5.2.** *For all  $f \in C^\infty(X, \mathbb{C})$ ,  $b \in \Theta_X^k$  and  $\mu \in \Omega_X^l$ , we have:*

$$\begin{aligned} d_\pi(fb) &= -(\pi^\sharp \partial f) \wedge b + f(d_\pi b), \\ \partial_\pi(f\mu) &= (\pi^\sharp \partial f) \lrcorner \mu + f(\partial_\pi \mu). \end{aligned}$$

As a consequence, we have

**Proposition 5.3.**

$$\tau \circ \partial_A^\nabla = \bar{\partial} \circ \tau \quad (16)$$

$$\tau \circ \partial_B^\nabla = (-1)^{k+l+1} \partial_\pi \circ \tau \quad (17)$$

*Proof.* The first relation (16) is a simple consequence of the definition (12) of  $\partial_A^\nabla$ , while the second (17) follows from (13), (15) and Lemma 5.2.  $\square$

Now the conclusion of the theorem follows immediately.  $\square$

## 6. EVENS-LU-WEINSTEIN DUALITY

We recall a remarkable duality construction due to Evens, Lu & Weinstein [7].

Consider a compact complex (and therefore orientable) manifold  $X$  with  $\dim_{\mathbb{C}} X = n$ , a complex Lie algebroid  $B$  over  $X$  with  $\text{rk}_{\mathbb{C}} B = r$  and a module  $E$  over  $B$ . The complex dual  $E^*$  is also a module over  $B$ . We will use the symbol  $\nabla$  to denote the representations of  $B$  on both  $E$  and  $E^*$ .

The complex vector bundle  $Q_B = \wedge^r B \otimes \wedge^{2n}(T_X \otimes \mathbb{C})^*$  is a module over the complex Lie algebroid  $B$  with representation  $D : \Gamma(Q_B) \rightarrow \Gamma(B^* \otimes Q_B)$  [7] given by

$$D_b(X \otimes \mu) = [b, X] \otimes \mu + X \otimes L_{\rho(b)}\mu,$$

for all  $b \in \Gamma(B)$ ,  $X \in \Gamma(\wedge^r B)$  and  $\mu \in \Gamma(\wedge^{2n}(T_X \otimes \mathbb{C})^*)$ .

By  $H^*(B, E)$  and  $H^*(B, E^* \otimes Q_B)$ , we denote the Lie algebroid cohomology of  $B$  with coefficients in  $E$  and  $E^* \otimes Q_B$ , respectively. We use the notation  $d_B^{\nabla}$  to denote their coboundary differential operators in both cases. Let  $\Xi$  be the isomorphism of vector bundles:

$$\Xi : \wedge^r B^* \otimes (\wedge^r B \otimes \wedge^{2n}(T_X \otimes \mathbb{C})^*) \rightarrow \wedge^{2n}(T_X \otimes \mathbb{C})^* : \xi \otimes (X \otimes \mu) \mapsto (\xi \lrcorner X)\mu.$$

The following lemma can be verified by a direct computation.

**Lemma 6.1.** *We have*

$$\Xi \circ d_B^{\nabla}(\xi \otimes (X \otimes \mu)) = (-1)^{r-1} d(\rho(\xi \lrcorner X) \lrcorner \mu),$$

for any  $\xi \otimes (X \otimes \mu) \in \Gamma(\wedge^{r-1} B^* \otimes Q_B)$

Consider the bilinear map

$$\lrcorner \cdot, \cdot \lrcorner : \Gamma(\wedge^k B^* \otimes E) \otimes \Gamma(\wedge^{r-k} B^* \otimes E^* \otimes Q_B) \rightarrow \Gamma(\wedge^{2n}(T_X \otimes \mathbb{C})^*)$$

defined by

$$\lrcorner \xi_1 \otimes e, \xi_2 \otimes \epsilon \otimes (X \otimes \mu) \lrcorner = \epsilon(e) \cdot (\xi_1 \wedge \xi_2)(X) \cdot \mu.$$

**Lemma 6.2.** *If  $\xi_1 \otimes e \in \Gamma(\wedge^{k-1} B^* \otimes E)$  and  $\xi_2 \otimes \epsilon \otimes (X \otimes \mu) \in \Gamma(\wedge^{r-k} B^* \otimes E^* \otimes Q_B)$ , then*

$$\begin{aligned} & \lrcorner d_B^{\nabla}(\xi_1 \otimes e), \xi_2 \otimes \epsilon \otimes (X \otimes \mu) \lrcorner + (-1)^{r-1} \lrcorner \xi_1 \otimes e, d_B^{\nabla}(\xi_2 \otimes \epsilon \otimes (X \otimes \mu)) \lrcorner \\ & = \Xi \circ d_B^{\nabla}(\epsilon(e) \cdot \xi \otimes (X \otimes \mu)) = (-1)^{r-1} d(\epsilon(e) \cdot \rho(\xi \lrcorner X) \lrcorner \mu). \end{aligned}$$

Therefore, by Stokes' theorem, the pairing

$$\langle \alpha, \beta \rangle = \int_X \lrcorner \alpha, \beta \lrcorner$$

where  $\alpha \in \Gamma(\wedge^k B^* \otimes E)$  and  $\beta \in \Gamma(\wedge^{r-k} B^* \otimes E^* \otimes Q_B)$  satisfies

$$\langle d_B^{\nabla}(\alpha), \beta \rangle + (-1)^{r-1} \langle \alpha, d_B^{\nabla}(\beta) \rangle = 0$$

(where  $\alpha \in \Gamma(\wedge^{k-1} B^* \otimes E)$  and  $\beta \in \Gamma(\wedge^{r-k} B^* \otimes E^* \otimes Q_B)$ ) and thus induces a pairing at the cohomology level [7]:

$$\langle \cdot, \cdot \rangle : H^k(B, E) \otimes H^{r-k}(B, E^* \otimes Q_B) \rightarrow \mathbb{C}. \quad (18)$$

The following is due to Block [1].

**Proposition 6.3.** *If  $B$  is an elliptic Lie algebroid, the cohomology pairing (18) is perfect.*

Given a holomorphic Poisson manifold  $(X, \pi)$ , we can take  $B = T_X^{0,1} \bowtie (T_X^{1,0})^*$ . Then  $Q_B^{\frac{1}{2}} = \wedge^n (T_X^{1,0})^*$  and, taking  $E = Q_B^{\frac{1}{2}}$ , we have  $E = \wedge^n (T_X^{1,0})^*$  and  $E^* \otimes Q_B = \wedge^n (T_X^{1,0})^*$ .

In this particular case, we get the cohomology pairing

$$\langle \cdot, \cdot \rangle : H^k(T_X^{0,1} \bowtie (T_X^{1,0})^*, \wedge^n (T_X^{1,0})^*) \otimes H^{2n-k}(T_X^{0,1} \bowtie (T_X^{1,0})^*, \wedge^n (T_X^{1,0})^*) \rightarrow \mathbb{C},$$

If we identify the cochain group  $\bigoplus_{k,l} C^{k,l}$  with  $\bigoplus_{p,q} \Omega_X^{p,q}$  via the contraction map  $\tau$  (see Equation (14)), then a straightforward (though lengthy) computation shows that, on the cochain level, the above cohomology pairing is given by

$$\Omega_X^{i,j} \otimes \Omega_X^{k,l} \rightarrow \mathbb{C} : \zeta \otimes \eta \mapsto \int_X (\zeta \wedge \eta)^{top}.$$

We have proved the following theorem.

**Theorem 6.4.** *Let  $(X, \pi)$  be a compact holomorphic Poisson manifold. The pairing*

$$\langle \cdot, \cdot \rangle : H_{2n-k}(X, \pi) \otimes H_k(X, \pi) \rightarrow \mathbb{C} : [\zeta] \otimes [\eta] \mapsto \int_X (\zeta \wedge \eta)^{top}$$

*(where  $\zeta, \eta \in \bigoplus_{k,l} \Omega_X^{k,l}$ ) is nondegenerate.*

**Remark 6.5.** *When  $X$  is a compact complex manifold considered as a zero Poisson manifold, then  $H_k(X, \pi) \cong \bigoplus_{j-i=n-k} H^{i,j}(X)$ . The above theorem easily follows from Serre duality.*

## 7. EXAMPLES

The purpose of this section is the computation of the Koszul-Brylinski Poisson homology of all Poisson structures with which  $\mathbb{CP}^1 \times \mathbb{CP}^1$  can be endowed.

From now on,  $X$  will denote the complex manifold  $\mathbb{CP}^1 \times \mathbb{CP}^1$ . Since  $X$  is 2-dimensional, all holomorphic bivector fields on it are automatically Poisson tensors. Thus the Poisson tensors on  $X$  form the complex vector space  $H^0(X, \wedge^2 T_X)$ , which is known to be 9-dimensional. Here is a more explicit description of  $H^0(X, \wedge^2 T_X)$ .

**Proposition 7.1.** [5] Let  $P^{2,2}$  denote the 9-dimensional vector space of all bihomogeneous polynomials on  $\mathbb{C}^2 \times \mathbb{C}^2$  of bidegree  $(2, 2)$ . Given any  $p \in P^{2,2}$ , there exists a unique holomorphic bivector field  $\pi_p$  on  $X = \mathbb{CP}^1 \times \mathbb{CP}^1$  such that, in an affine chart  $(z_1, z_2) \mapsto ([1 : z_1], [1 : z_2])$  of  $\mathbb{CP}^1 \times \mathbb{CP}^1$ , we have

$$\pi_p = q(z_1, z_2) \partial_{z_2} \wedge \partial_{z_1},$$

where  $q(z_1, z_2) = p((1, z_1), (1, z_2))$ . The map

$$p \in P^{2,2} \longmapsto \pi_p \in H^0(X, \wedge^2 T_X)$$

is an isomorphism of complex vector spaces. As a consequence, the space of all holomorphic Poisson bivector fields is a 9-dimensional vector space over  $\mathbb{C}$ .

**Theorem 7.2.** For any holomorphic Poisson bivector field  $\pi$  on  $X = \mathbb{CP}^1 \times \mathbb{CP}^1$ , we have

$$H_0(X, \pi) = 0, \quad H_1(X, \pi) = 0, \quad H_2(X, \pi) \cong \mathbb{C}^4, \quad H_3(X, \pi) = 0, \quad H_4(X, \pi) = 0.$$

*Proof.* Let us first assume that  $\pi = 0$ . In this case,  $H_k(X, \pi) = \bigoplus_{j-i=2-k} H^{i,j}(X)$  and we obtain

$$\begin{aligned} H_0(X, \pi) &= H^{0,2}(X) = 0, & H_1(X, \pi) &= H^{1,2}(X) \oplus H^{0,1}(X) = 0, \\ H_3(X, \pi) &= H^{2,0}(X) = 0, & H_4(X, \pi) &= H^{2,1}(X) \oplus H^{1,0}(X) = 0, \end{aligned}$$

and

$$H_2(X, \pi) = H^{0,0}(X) \oplus H^{1,1}(X) \oplus H^{2,2}(X) \cong \mathbb{C}^4.$$

Now let us assume that  $\pi \neq 0$ . By definition,  $H_0(X, \pi)$  consists of those  $\alpha \in \Omega_X^{2,0}$  such that  $\bar{\partial}\alpha = 0$  and  $\partial_\pi\alpha = 0$ . The first condition means that  $\alpha$  is a holomorphic 2-form on  $X$ . Since  $H^0(X, \Omega_X^2) = H^{0,2}(X) = H^{0,2}(\mathbb{CP}^1 \times \mathbb{CP}^1) = 0$ , it follows that  $H_0(X, \pi) = 0$ .

We now proceed to compute  $H_1(X, \pi)$ . Assume that  $\theta + \omega \in \Omega^{1,0} \oplus \Omega^{2,1}$  is a Koszul-Brylinski 1-cycle. That is,  $\partial_\pi\theta + \bar{\partial}\theta + \partial_\pi\omega + \bar{\partial}\omega = 0$ . Hence it follows that  $\partial_\pi\theta = 0$ ,  $\bar{\partial}\theta + \partial_\pi\omega = 0$  and  $\bar{\partial}\omega = 0$ . Since  $H^{1,2}(\mathbb{CP}^1 \times \mathbb{CP}^1) = 0$ , there exists  $\beta \in \Omega_X^{2,0}$  with  $\omega = \bar{\partial}\beta$ . On the other hand, from  $\partial_\pi\theta = 0$ , it follows that  $i_\pi\bar{\partial}\theta = 0$  since  $\partial_\pi\theta = [\partial, i_\pi]\theta = -i_\pi\bar{\partial}\theta$ . Therefore,  $\bar{\partial}\theta$  vanishes at those points where  $\pi$  does not vanish. By Proposition 7.1,  $\pi$  is nonzero on a dense subset of  $\mathbb{CP}^1 \times \mathbb{CP}^1$ . Thus, we have  $\bar{\partial}\theta = 0$ , which implies that  $\theta = \partial\alpha$  for some  $\alpha \in \Omega_X^{0,0}$  since  $H^{1,0}(\mathbb{CP}^1 \times \mathbb{CP}^1) = 0$ . It follows that

$$0 = \bar{\partial}\theta + \partial_\pi\omega = \bar{\partial}\partial\alpha + \partial_\pi\bar{\partial}\beta = \bar{\partial}(-\partial\alpha - \partial_\pi\beta).$$

Since  $H^{0,1}(\mathbb{CP}^1 \times \mathbb{CP}^1) = 0$ , we have  $\partial\alpha + \partial_\pi\beta = 0$ . Thus  $\theta + \omega = (\bar{\partial} - \partial_\pi)\beta$  from which we conclude that  $H_1(X, \pi) = 0$ .

By Evens-Lu-Weinstein duality, we have

$$H_3(X, \pi) \cong H_1(X, \pi) = 0 \quad \text{and} \quad H_4(X, \pi) \cong H_0(X, \pi) = 0.$$

Moreover, according to Theorem 4.5,

$$\chi_{KB}(X, \pi) = \chi(X) = \chi(\mathbb{CP}^1) + \chi(\mathbb{CP}^1) = 4.$$

Thus we have  $H_2(X, \pi) \cong \mathbb{C}^4$ . This concludes the proof.  $\square$

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